

L_2 - and $S_{p,q}^r B$ -discrepancy of (order 2) digital nets

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Dick proved that all order 2 digital nets satisfy optimal upper bounds of the L_2 -discrepancy. We give an alternative proof for this fact using Haar bases. Furthermore, we prove that all digital nets satisfy optimal upper bounds of the $S_{p,q}^r B$ -discrepancy for a certain parameter range and enlarge that range for order 2 digital nets. L_p -, $S_{p,q}^r F$ - and $S_p^r H$ -discrepancy is considered as well.

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1 Introduction and results

Let N be some positive integer and let \mathcal{P} be a point set in the unit cube $[0, 1)^d$ with N points. Then the discrepancy function $D_{\mathcal{P}}$ is defined as

$$D_{\mathcal{P}}(x) = \frac{1}{N} \sum_{z \in \mathcal{P}} \chi_{[0,x)}(z) - x_1 \cdot \dots \cdot x_d. \quad (1)$$

for any $x = (x_1, \dots, x_d) \in [0, 1)^d$. By $\chi_{[0,x)}$ we mean the characteristic function of the interval $[0, x) = [0, x_1) \times \dots \times [0, x_d)$, so the term $\sum_z \chi_{[0,x)}(z)$ is equal to $\#(\mathcal{P} \cap [0, x))$.

This means that $D_{\mathcal{P}}$ measures the deviation of the number of points of \mathcal{P} in $[0, x)$ from the fair number of points $N|[0, x)| = N x_1 \cdot \dots \cdot x_d$ which would be achieved by a (practically impossible) perfectly uniform distribution of the points of \mathcal{P} , normalized by the total number of points.

Usually one is interested in calculating the norm of the discrepancy function in some normed space of functions on $[0, 1)^d$ to which the discrepancy function belongs. A well known result concerns $L_p([0, 1)^d)$ -spaces for $1 < p < \infty$. There exists a constant $c_{p,d} > 0$ such that for every positive integer N and all point sets \mathcal{P} in $[0, 1)^d$ with N points, we have

$$\|D_{\mathcal{P}}|_{L_p([0, 1)^d)}\| \geq c_{p,d} \frac{(\log N)^{(d-1)/2}}{N}. \quad (2)$$

It was proved by Roth [R54] for $p = 2$ and by Schmidt [S77] for arbitrary $1 < p < \infty$. The best value for $c_{2,d}$ can be found in [HM11]. Furthermore, there exists a constant $C_{p,d} > 0$ such that for every positive integer N , there exists a point set \mathcal{P} in $[0, 1)^d$ with N points such that

$$\|D_{\mathcal{P}}|_{L_p(\mathbb{Q}^d)}\| \leq C_{p,d} \frac{(\log N)^{(d-1)/2}}{N}. \quad (3)$$

It was proved by Davenport [D56] for $p = 2, d = 2$, by Roth [R80] for $p = 2$ and arbitrary d and finally by Chen [C80] in the general case. The best value for $C_{2,d}$ can be found in [DP10] and [FPPS10].

There are results for $L_1([0, 1)^d)$ - and star ($L_\infty([0, 1)^d)$ -) discrepancy though there are still gaps between lower and upper bounds, see [H81], [S72], [BLV08]. As general references for studies of the discrepancy function we refer to the monographs [DP10], [NW10], [M99], [KN74] and surveys [B11], [Hi14], [M13c].

Roth's and Chen's original proofs of (3) were probabilistic. Explicit constructions of point sets with good L_p -discrepancy in arbitrary dimension have not been known for a long time. Chen and Skriganov [CS02] (see also [CS08] and [DP10]) gave constructions with optimal bound of the L_2 -discrepancy and Skriganov [S06] later proved the L_p bound. The constructions of Chen and Skriganov were order 1 digital nets with large Hamming weight. Dick and Pillichshammer [DP14a] (see also [DP14b]) gave alternative constructions. Their constructions are order 3 digital nets. Dick [D14] proved then the following result.

Theorem 1.1. *There exists a constant $C_{d,b,v} > 0$ such that for every positive integer n*

and every order 2 digital (v, n, d) -net \mathcal{P}_n^b in base b we have

$$\left\| D_{\mathcal{P}_n^b} |L_2([0, 1)^d) \right\| \leq C_{d,b,v} \frac{n^{(d-1)/2}}{b^n}.$$

In this work we give an alternative proof for this fact.

Furthermore, there are results for the discrepancy in other function spaces, like Hardy spaces, logarithmic and exponential Orlicz spaces, weighted L_p -spaces, BMO (see [B11] for results and further literature).

Here, we are interested in Besov $(S_{p,q}^r B([0, 1)^d))$, Triebel-Lizorkin $(S_{p,q}^r F([0, 1)^d))$ and Sobolev $(S_p^r H([0, 1)^d))$ spaces with dominating mixed smoothness. Triebel [T10] proved that for all $1 \leq p, q \leq \infty$ with $q < \infty$ if $p = \infty$ and all $r \in \mathbb{R}$ satisfying $1/p - 1 < r < 1/p$, there exists a constant $c_{p,q,r,d} > 0$ such that for every integer $N \geq 2$ and all point sets \mathcal{P} in $[0, 1)^d$ with N points, we have

$$\left\| D_{\mathcal{P}} |S_{p,q}^r B([0, 1)^d) \right\| \geq c_{p,q,r,d} N^{r-1} (\log N)^{(d-1)/q} \quad (4)$$

and with the additional condition that $q > 1$ if $p = \infty$ there exists a constant $C_{p,q,r,d} > 0$ such that for every positive integer N , there exists a point set \mathcal{P} in $[0, 1)^d$ with N points and we have

$$\left\| D_{\mathcal{P}} |S_{p,q}^r B([0, 1)^d) \right\| \leq C_{p,q,r,d} N^{r-1} (\log N)^{(d-1)(1/q+1-r)}.$$

Hinrichs [Hi10] proved for $d = 2$ that for all $1 \leq p, q \leq \infty$ and all $0 \leq r < 1/p$ there exists a constant $C_{p,q,r} > 0$ such that for every integer $N \geq 2$ there exists a point set \mathcal{P} in $[0, 1)^2$ with N points such that

$$\left\| D_{\mathcal{P}} |S_{p,q}^r B([0, 1)^2) \right\| \leq C_{p,q,r} N^{r-1} (\log N)^{1/q}.$$

Markhasin [M13b] proved that for all $1 \leq p, q \leq \infty$ and all $0 < r < 1/p$ there exists a constant $C_{p,q,r,d} > 0$ such that for every integer $N \geq 2$ there exists a point set \mathcal{P} in $[0, 1)^d$ with N points such that

$$\left\| D_{\mathcal{P}} |S_{p,q}^r B([0, 1)^d) \right\| \leq C_{p,q,r,d} N^{r-1} (\log N)^{(d-1)/q}. \quad (5)$$

Explicit point sets with optimal bounds of $S_{p,q}^r B$ -discrepancy used in [M13b] are the already mentioned point sets by Chen and Skriganov. In $d = 2$ also (generalized) Hammersley point sets can be used (see [Hi10], [M13a]). Our goal is to prove that there are way more point sets with optimal bounds of the $S_{p,q}^r B$ -discrepancy. Furthermore

there are results for spaces $S_{p,q}^r F([0,1]^d)$ and $S_p^r H([0,1]^d)$ in [M13c].

Theorem 1.2. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $0 < r < 1/p$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every integer n and every order 1 digital (v,n,d) -net \mathcal{P}_n^b in base b we have*

$$\left\| D_{\mathcal{P}_n^b} |S_{p,q}^r B([0,1]^d) \right\| \leq C_{p,q,r,d,b,v} b^{n(r-1)} n^{(d-1)/q}.$$

Theorem 1.3. *Let $1 \leq p, q \leq \infty$, ($q > 1$ if $p = \infty$) and $0 \leq r < 1/p$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every positive integer n and every order 2 digital (v,n,d) -net \mathcal{P}_n^b in base b we have*

$$\left\| D_{\mathcal{P}_n^b} |S_{p,q}^r B([0,1]^d) \right\| \leq C_{p,q,r,d,b,v} b^{n(r-1)} n^{(d-1)/q}.$$

Corollary 1.4. *Let $1 \leq p, q < \infty$ and $0 < r < 1/\max(p,q)$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every positive integer n and every order 1 digital (v,n,d) -net \mathcal{P}_n^b in base b we have*

$$\left\| D_{\mathcal{P}_n^b} |S_{p,q}^r F([0,1]^d) \right\| \leq C_{p,q,r,d,b,v} b^{n(r-1)} n^{(d-1)/q}.$$

Corollary 1.5. *Let $1 \leq p, q < \infty$ and $0 \leq r < 1/\max(p,q)$. There exists a constant $C_{p,q,r,d,b,v} > 0$ such that for every positive integer n and every order 2 digital (v,n,d) -net \mathcal{P}_n^b in base b we have*

$$\left\| D_{\mathcal{P}_n^b} |S_{p,q}^r F([0,1]^d) \right\| \leq C_{p,q,r,d,b,v} b^{n(r-1)} n^{(d-1)/q}.$$

Corollary 1.6. *Let $1 \leq p < \infty$ and $0 < r < 1/\max(p,2)$. There exists a constant $C_{p,r,d,b,v} > 0$ such that for every positive integer n and every order 1 digital (v,n,d) -net \mathcal{P}_n^b in base b we have*

$$\left\| D_{\mathcal{P}_n^b} |S_p^r H([0,1]^d) \right\| \leq C_{p,r,d,b,v} b^{n(r-1)} n^{(d-1)/2}.$$

Corollary 1.7. *Let $1 \leq p < \infty$ and $0 \leq r < 1/\max(p,2)$. There exists a constant $C_{p,r,d,b,v} > 0$ such that for every positive integer n and every order 2 digital (v,n,d) -net \mathcal{P}_n^b in base b we have*

$$\left\| D_{\mathcal{P}_n^b} |S_p^r H([0,1]^d) \right\| \leq C_{p,r,d,b,v} b^{n(r-1)} n^{(d-1)/2}.$$

Theorem 1.8. *Let $1 \leq p < \infty$. There exists a constant $C_{p,d,b,v} > 0$ such that for every*

positive integer n and every order 2 digital (v, n, d) -net \mathcal{P}_n^b in base b we have

$$\left\| D_{\mathcal{P}_n^b} |L_p([0, 1]^d) \right\| \leq C_{p,d,b,v} \frac{n^{(d-1)/2}}{b^n}.$$

We point out that obviously Theorem 1.1 is a consequence of Theorem 1.8. Nevertheless, we will prove them independently, so that readers without a background in function spaces with dominating mixed smoothness (which is required for the proof of Theorem 1.8) will be able to understand the proof of the L_2 bound.

Theorems 1.2 and 1.3 are consistent with older results. Chen-Skriganov point sets are order 1 digital (v, n, d) -nets while (generalized) Hammersley point sets are order 2 digital $(0, n, 2)$ -nets

2 Function spaces with dominating mixed smoothness

We define the spaces $S_{p,q}^r B([0, 1]^d)$, $S_{p,q}^r F([0, 1]^d)$ and $S_p^r H([0, 1]^d)$ according to [T10]. Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions on \mathbb{R}^d . Let $\varphi_0 \in \mathcal{S}(\mathbb{R})$ satisfy $\varphi_0(x) = 1$ for $|x| \leq 1$ and $\varphi_0(x) = 0$ for $|x| > \frac{3}{2}$. Let $\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x)$ where $x \in \mathbb{R}$, $k \in \mathbb{N}$ and $\varphi_k(x) = \varphi_{k_1}(x_1) \dots \varphi_{k_d}(x_d)$ where $k = (k_1, \dots, k_d) \in \mathbb{N}_0^d$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. The functions φ_k are a dyadic resolution of unity since

$$\sum_{k \in \mathbb{N}_0^d} \varphi_k(x) = 1$$

for all $x \in \mathbb{R}^d$. The functions $\mathcal{F}^{-1}(\varphi_k \mathcal{F}f)$ are entire analytic functions for every $f \in \mathcal{S}'(\mathbb{R}^d)$.

Let $0 < p, q \leq \infty$ and $r \in \mathbb{R}$. The Besov space with dominating mixed smoothness $S_{pq}^r B(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ with finite quasi-norm

$$\left\| f |S_{pq}^r B(\mathbb{R}^d) \right\| = \left(\sum_{k \in \mathbb{N}_0^d} 2^{r(k_1 + \dots + k_d)q} \left\| \mathcal{F}^{-1}(\varphi_k \mathcal{F}f) |L_p(\mathbb{R}^d) \right\|^q \right)^{\frac{1}{q}} \quad (6)$$

with the usual modification if $q = \infty$.

Let $0 < p < \infty$, $0 < q \leq \infty$ and $r \in \mathbb{R}$. The Triebel-Lizorkin space with dominating

mixed smoothness $S_{pq}^r F(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ with finite quasi-norm

$$\|f|_{S_{pq}^r F(\mathbb{R}^d)}\| = \left\| \left(\sum_{k \in \mathbb{N}_0^d} 2^{r(k_1 + \dots + k_d)q} \left| \mathcal{F}^{-1}(\varphi_k \mathcal{F}f)(\cdot) \right|^q \right)^{\frac{1}{q}} \right\|_{L_p(\mathbb{R}^d)} \quad (7)$$

with the usual modification if $q = \infty$.

Let $\mathcal{D}([0, 1]^d)$ consist of all complex-valued infinitely differentiable functions on \mathbb{R}^d with compact support in the interior of $[0, 1]^d$ and let $\mathcal{D}'([0, 1]^d)$ be its dual space of all distributions in $[0, 1]^d$. The Besov space with dominating mixed smoothness $S_{pq}^r B([0, 1]^d)$ consists of all $f \in \mathcal{D}'([0, 1]^d)$ with finite quasi-norm

$$\|f|_{S_{pq}^r B([0, 1]^d)}\| = \inf \left\{ \|g|_{S_{pq}^r B(\mathbb{R}^d)}\| : g \in S_{pq}^r B(\mathbb{R}^d), g|_{[0, 1]^d} = f \right\}. \quad (8)$$

The Triebel-Lizorkin space with dominating mixed smoothness $S_{pq}^r F([0, 1]^d)$ consists of all $f \in \mathcal{D}'([0, 1]^d)$ with finite quasi-norm

$$\|f|_{S_{pq}^r F([0, 1]^d)}\| = \inf \left\{ \|g|_{S_{pq}^r F(\mathbb{R}^d)}\| : g \in S_{pq}^r F(\mathbb{R}^d), g|_{[0, 1]^d} = f \right\}. \quad (9)$$

The spaces $S_{pq}^r B(\mathbb{R}^d)$, $S_{pq}^r F(\mathbb{R}^d)$, $S_{pq}^r B([0, 1]^d)$ and $S_{pq}^r F([0, 1]^d)$ are quasi-Banach spaces. We define the Sobolev space with dominating mixed smoothness as

$$S_p^r H([0, 1]^d) = S_{p2}^r F([0, 1]^d). \quad (10)$$

If $r \in \mathbb{N}_0$ then it is denoted by $S_p^r W([0, 1]^d)$ and is called classical Sobolev space with dominating mixed smoothness. An equivalent norm for $S_p^r W([0, 1]^d)$ is

$$\sum_{\alpha \in \mathbb{N}_0^d; 0 \leq \alpha_i \leq r} \|D^\alpha f|_{L_p([0, 1]^d)}\|.$$

Of special interest is the case $r = 0$ since

$$S_p^0 H([0, 1]^d) = L_p([0, 1]^d).$$

The Besov and Triebel-Lizorkin spaces can be embedded in each other (see [T10] or [M13c, Corollary 1.13]). We point out that the following embedding is a combination of well known results and might look odd at the first glance.

Lemma 2.1. *Let $0 < p, q < \infty$ and $r \in \mathbb{R}$. Then we have*

$$S_{\max(p,q),q}^r B([0,1]^d) \hookrightarrow S_{pq}^r F([0,1]^d) \hookrightarrow S_{\min(p,q),q}^r B([0,1]^d).$$

3 Haar and Walsh bases

We denote $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$. Let $b \geq 2$ be an integer. We denote $\mathbb{D}_j = \{0, 1, \dots, b^j - 1\}$ and $\mathbb{B}_j = \{1, \dots, b - 1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} = \{0\}$ and $\mathbb{B}_{-1} = \{1\}$. For $j = (j_1, \dots, j_d) \in \mathbb{N}_{-1}^d$ let $\mathbb{D}_j = \mathbb{D}_{j_1} \times \dots \times \mathbb{D}_{j_d}$ and $\mathbb{B}_j = \mathbb{B}_{j_1} \times \dots \times \mathbb{B}_{j_d}$. For a real a we write $a_+ = \max(a, 0)$ and for $j \in \mathbb{N}_{-1}^d$ we write $|j|_+ = j_{1+} + \dots + j_{d+}$.

For $j \in \mathbb{N}_0$ and $m \in \mathbb{D}_j$ we call the interval

$$I_{j,m} = [b^{-j}m, b^{-j}(m+1))$$

the m -th b -adic interval in $[0, 1)$ on level j . We put $I_{-1,0} = [0, 1)$ and call it the 0-th b -adic interval in $[0, 1)$ on level -1 . For any $k = 0, \dots, b - 1$ let $I_{j,m}^k = I_{j+1, bm+k}$. We put $I_{-1,0}^{-1} = I_{-1,0} = [0, 1)$. For $j \in \mathbb{N}_{-1}^d$ and $m = (m_1, \dots, m_d) \in \mathbb{D}_j$ we call

$$I_{j,m} = I_{j_1, m_1} \times \dots \times I_{j_d, m_d}$$

the m -th b -adic interval in $[0, 1)^d$ on level j . We call the number $|j|_+$ the order of the b -adic interval $I_{j,m}$. Its volume is $b^{-|j|_+}$.

Let $j \in \mathbb{N}_0$, $m \in \mathbb{D}_j$ and $l \in \mathbb{B}_j$. Let $h_{j,m,l}$ be the function on $[0, 1)$ with support in $I_{j,m}$ and the constant value $e^{\frac{2\pi i}{b}lk}$ on $I_{j,m}^k$ for any $k = 0, \dots, b - 1$. We put $h_{-1,0,1} = \chi_{I_{-1,0}}$ on $[0, 1)$.

Let $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$ and $l = (l_1, \dots, l_d) \in \mathbb{B}_j$. The function $h_{j,m,l}$ given as the tensor product

$$h_{j,m,l}(x) = h_{j_1, m_1, l_1}(x_1) \dots h_{j_d, m_d, l_d}(x_d)$$

for $x = (x_1, \dots, x_d) \in [0, 1)^d$ is called a b -adic Haar function on $[0, 1)^d$. The functions $h_{j,m,l}$, $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$ are called b -adic Haar basis on $[0, 1)^d$.

The following result is [M13c, Theorem 2.1].

Theorem 3.1. *The system*

$$\left\{ b^{\frac{|j|_+}{2}} h_{j,m,l} : j \in \mathbb{N}_{-1}^d, m \in \mathbb{D}_j, l \in \mathbb{B}_j \right\}$$

is an orthonormal basis of $L_2([0, 1)^d)$, an unconditional basis of $L_p([0, 1)^d)$ for $1 < p < \infty$

and a conditional basis of $L_1([0, 1]^d)$. For any function $f \in L_2([0, 1]^d)$ we have

$$\|f|L_2([0, 1]^d)\|^2 = \sum_{j \in \mathbb{N}_{-1}^d} b^{|j|} \sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle f, h_{j,m,l} \rangle|^2.$$

The following result is [M13c, Theorem 2.11].

Theorem 3.2. *Let $0 < p, q \leq \infty$, ($q > 1$ if $p = \infty$) and $1/p - 1 < r < \min(1/p, 1)$. Let $f \in \mathcal{D}'([0, 1]^d)$. Then $f \in S_{pq}^r B([0, 1]^d)$ if and only if it can be represented as*

$$f = \sum_{j \in \mathbb{N}_{-1}^d} b^{|j|+} \sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} \mu_{j,m,l} h_{j,m,l} \quad (11)$$

for some sequence $(\mu_{j,m,l})$ satisfying

$$\left(\sum_{j \in \mathbb{N}_{-1}^d} b^{|j|+(r-1/p+1)q} \left(\sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\mu_{j,m,l}|^p \right)^{q/p} \right)^{1/q} < \infty. \quad (12)$$

The convergence of (11) is unconditional in $\mathcal{D}'([0, 1]^d)$ and in any $S_{pq}^\rho B([0, 1]^d)$ with $\rho < r$. The representation (11) of f is unique with the b -adic Haar coefficients $\mu_{j,m,l} = \langle f, h_{j,m,l} \rangle$. The expression (12) is an equivalent quasi-norm on $S_{pq}^r B([0, 1]^d)$.

For $\alpha \in \mathbb{N}$ with the b -adic expansion $\alpha = \beta_{a_1-1}b^{a_1-1} + \dots + \beta_{a_\nu-1}b^{a_\nu-1}$ with $0 < a_1 < a_2 < \dots < a_\nu$ and digits $\beta_{a_1-1}, \dots, \beta_{a_\nu-1} \in \{1, \dots, b-1\}$, the NRT weight of order $\sigma \in \mathbb{N}$ is given by

$$\varrho_\sigma(\alpha) = a_\nu + a_{\nu-1} + \dots + a_{\max(\nu-\sigma+1, 1)}.$$

Furthermore, $\varrho_\sigma(0) = 0$.

For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, the NRT weight of order σ is given by

$$\varrho_\sigma(\alpha) = \varrho_\sigma(\alpha_1) + \dots + \varrho_\sigma(\alpha_d).$$

Let $\alpha \in \mathbb{N}$. The α -th b -adic Walsh function $\text{wal}_\alpha : [0, 1] \rightarrow \mathbb{C}$ is given by

$$\text{wal}_\alpha(x) = e^{\frac{2\pi i}{b}(\beta_{a_1-1}x_{a_1} + \dots + \beta_{a_\nu-1}x_{a_\nu})}$$

for $x \in [0, 1]$ with b -adic expansion $x = x_1b^{-1} + x_2b^{-2} + \dots$. Furthermore, $\text{wal}_0 = \chi_{[0,1]}$.

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. Then the α -th b -adic Walsh function wal_α on $[0, 1]^d$ is

given as the tensor product

$$\text{wal}_\alpha(x) = \text{wal}_{\alpha_1}(x^1) \dots \text{wal}_{\alpha_d}(x^d)$$

for $x = (x^1, \dots, x^d) \in [0, 1]^d$. The functions wal_α , $\alpha \in \mathbb{N}_0^d$ are called b -adic Walsh basis on $[0, 1]^d$.

The b -adic Walsh function wal_α is constant on b -adic intervals $I_{(\varrho_1(\alpha_1), \dots, (\varrho_1(\alpha_d)), m}$ for every $m \in \mathbb{D}_{(\varrho_1(\alpha_1), \dots, (\varrho_1(\alpha_d))}$. The following result is [DP10, Theorem A.11].

Lemma 3.3. *The system*

$$\left\{ \text{wal}_\alpha : \alpha \in \mathbb{N}_0^d \right\}$$

is an orthonormal basis of $L_2([0, 1]^d)$.

4 Digital (v, n, d) -nets

We quote from [DP14a] and [D07] to describe the digital construction method and properties of resulting digital nets. We also refer to [N87] and [NP01].

For an integer $b \geq 2$ let \mathbb{Z}_b denote the commutative ring of integers modulo b . For $s, n \in \mathbb{N}$ with $s \geq n$ let C_1, \dots, C_d be $s \times n$ matrices with entries from \mathbb{Z}_b . For $\nu \in \{0, 1, \dots, b^n - 1\}$ with the b -adic expansion $\nu = \nu_0 + \nu_1 b + \dots + \nu_{n-1} b^{n-1}$ with digits $\nu_0, \nu_1, \dots, \nu_{n-1} \in \{0, 1, \dots, b-1\}$ the b -adic digit vector $\bar{\nu}$ is given as $\bar{\nu} = (\nu_0, \nu_1, \dots, \nu_{n-1})^\top \in \mathbb{Z}_b^n$. Then we compute $C_i \bar{\nu} = (x_{i,\nu,1}, x_{i,\nu,2}, \dots, x_{i,\nu,s})^\top \in \mathbb{Z}_b^s$ for $1 \leq i \leq d$. Finally we define

$$x_{i,\nu} = x_{i,\nu,1} b^{-1} + x_{i,\nu,2} b^{-2} + \dots + x_{i,\nu,s} b^{-s} \in [0, 1)$$

and $x_\nu = (x_{1,\nu}, \dots, x_{d,\nu})$. We call the point set $\mathcal{P}_n^b = \{x_0, x_1, \dots, x_{b^n-1}\}$ a digital net in base b .

Now let $\sigma \in \mathbb{N}$ and suppose $s \geq \sigma n$. Let $0 \leq v \leq \sigma n$ be an integer. For every $1 \leq i \leq d$ we write $C_i = (c_{i,1}, \dots, c_{i,s})^\top$ where $c_{i,1}, \dots, c_{i,s} \in \mathcal{P}_b^n$ are the row vectors of C_i . If for all $1 \leq \lambda_{i,1} < \dots < \lambda_{i,\eta_i} \leq s$, $1 \leq i \leq d$ with

$$\lambda_{1,1} + \dots + \lambda_{1,\min(\eta_1,\sigma)} + \dots + \lambda_{d,1} + \dots + \lambda_{d,\min(\eta_d,\sigma)} \leq \sigma n - v$$

the vectors $c_{1,\lambda_{1,1}}, \dots, c_{1,\lambda_{1,\eta_1}}, \dots, c_{d,\lambda_{d,1}}, \dots, c_{d,\lambda_{d,\eta_d}}$ are linearly independent over \mathbb{Z}_b , then \mathcal{P}_n^b is called an order σ digital (v, n, d) -net in base b .

Lemma 4.1.

- (i) Let $v < \sigma n$. Then every order σ digital (v, n, d) -net in base b is an order σ digital $(v + 1, n, d)$ -net in base b . In particular every point set \mathcal{P}_n^b constructed with the digital method is at least an order σ digital $(\sigma n, n, d)$ -net in base b .
- (ii) Let $1 \leq \sigma_1 \leq \sigma_2$. Then every order σ_2 digital (v, n, d) -net in base b is an order σ_1 digital $(\lceil v\sigma_1/\sigma_2 \rceil, n, d)$ -net in base b .

Lemma 4.2. Let \mathcal{P}_n^b be an order σ digital (v, n, d) -net in base b then every b -adic interval of order $n - v$ contains exactly b^v points of \mathcal{P}_n^b .

Let $t \in \mathbb{N}_0$ with b -adic expansion $t = \tau_0 + \tau_1 b + \tau_2 b^2 + \dots$. We put $\bar{t} = (\tau_0, \tau_1, \dots, \tau_{s-1})^\top \in \mathbb{Z}_b^s$ and define

$$\mathfrak{D}(\mathfrak{C}) = \left\{ t = (t_1, \dots, t_d) \in \mathbb{N}_0^d \setminus \{(0, \dots, 0)\} : C_1^\top \bar{t}_1 + \dots + C_d^\top \bar{t}_d = (0, \dots, 0) \in \mathbb{Z}_b^n \right\}.$$

Lemma 4.3. \mathcal{P}_n^b is an order σ digital (v, n, d) -net in base b if and only if $\varrho_\sigma(t) > \sigma n - v$ for all $t \in \mathfrak{D}(\mathfrak{C})$.

Lemma 4.4. Let \mathcal{P}_n^b be an order σ digital (v, n, d) -net in base b with generating matrices C_1, \dots, C_d . Then

$$\sum_{z \in \mathcal{P}_n^b} \text{wal}_t(z) = \begin{cases} b^n & \text{if } t \in \mathfrak{D}(\mathfrak{C}), \\ 0 & \text{otherwise.} \end{cases}$$

We consider the Walsh series expansion of the function $\chi_{[0, x)}$,

$$\chi_{[0, x)}(y) = \sum_{t=0}^{\infty} \hat{\chi}_{[0, x)}(t) \text{wal}_t(y), \quad (13)$$

where for $t \in \mathbb{N}_0$ the t -th Walsh coefficient is given by

$$\hat{\chi}_{[0, x)}(t) = \int_0^1 \chi_{[0, x)}(y) \overline{\text{wal}_t(y)} dy = \int_0^x \overline{\text{wal}_t(y)} dy.$$

Lemma 4.5. Let \mathcal{P}_n^b be an order σ digital (v, n, d) -net in base b with generating matrices C_1, \dots, C_d . Then

$$D_{\mathcal{P}_n^b}(x) = \sum_{t \in \mathfrak{D}(\mathfrak{C})} \hat{\chi}_{[0, x)}(t).$$

Proof. For $t = (t_1, \dots, t_d) \in \mathbb{N}_0^d$ and $x = (x_1, \dots, x_d) \in [0, 1)^d$, we have

$$\hat{\chi}_{[0, x)}(t) = \hat{\chi}_{[0, x_1)}(t_1) \cdot \dots \cdot \hat{\chi}_{[0, x_d)}(t_d).$$

Applying Lemma 4.4 we get

$$\begin{aligned}
D_{\mathcal{P}}(x) &= \frac{1}{b^n} \sum_{z \in \mathcal{P}_n^b} \sum_{t_1, \dots, t_d=0}^{\infty} \hat{\chi}_{[0,x)}(t) \text{wal}_t(z) - \hat{\chi}_{[0,x)}((0, \dots, 0)) \\
&= \sum_{\substack{t_1, \dots, t_d=0 \\ (t_1, \dots, t_d) \neq (0, \dots, 0)}}^{\infty} \hat{\chi}_{[0,x)}(t) \frac{1}{b^n} \sum_{z \in \mathcal{P}} \text{wal}_t(z) \\
&= \sum_{t \in \mathfrak{D}(\mathfrak{C})} \hat{\chi}_{[0,x)}(t).
\end{aligned}$$

□

Several constructions of order σ digital (v, n, d) -nets are known. For details, examples and further literature we refer to [DP14b]. There are especially constructions with a good quality parameter v , e. g. we can construct order 2 digital (d, n, d) -nets in base b as well as order 1 digital $(0, n, d)$ -nets.

5 Proofs of the results

For two sequences a_n and b_n we will write $a_n \preceq b_n$ if there exists a constant $c > 0$ such that $a_n \leq c b_n$ for all n . For $t > 0$ with b -adic expansion $t = \tau_0 + \tau_1 b + \dots + \tau_{\varrho_1(t)-1} b^{\varrho_1(t)-1}$, we put $t = t' + \tau_{\varrho_1(t)-1} b^{\varrho_1(t)-1}$.

The following result is [M13b, Lemma 5.1].

Lemma 5.1. *Let $f(x) = x_1 \cdot \dots \cdot x_d$ for $x = (x_1, \dots, x_d) \in [0, 1)^d$. Let $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$. Then $|\langle f, h_{j,m,l} \rangle| \preceq b^{-2|j|_+}$.*

The following result is [M13b, Lemma 5.2].

Lemma 5.2. *Let $z = (z_1, \dots, z_d) \in [0, 1)^d$ and $g(x) = \chi_{[0,x)}(z)$ for $x = (x_1, \dots, x_d) \in [0, 1)^d$. Let $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$. Then $\langle g, h_{j,m,l} \rangle = 0$ if z is not contained in the interior of the b -adic interval $I_{j,m}$. If z is contained in the interior of $I_{j,m}$ then $|\langle g, h_{j,m,l} \rangle| \preceq b^{-|j|_+}$.*

The following result is [M13b, Lemma 5.9].

Lemma 5.3. *Let $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$ and $\alpha \in \mathbb{N}_0^d$. Then*

$$|\langle h_{j,m,l}, \text{wal}_\alpha \rangle| \preceq b^{-|j|_+}.$$

If $\varrho_1(\alpha_i) \neq j_i + 1$ for some $1 \leq i \leq d$ then

$$\langle h_{j,m,l}, \text{wal}_\alpha \rangle = 0.$$

The following result is [M13b, Lemma 5.10].

Lemma 5.4. *Let $t, \alpha \in \mathbb{N}_0$. Then*

$$|\langle \hat{\chi}_{[0,\cdot)}(t), \text{wal}_\alpha \rangle| \leq b^{-\max(\varrho_1(t), \varrho_1(\alpha))}.$$

If $\alpha \neq t'$ and $\alpha \neq t$ and $\alpha' \neq t$ then

$$\langle \hat{\chi}_{[0,\cdot)}(t), \text{wal}_\alpha \rangle = 0.$$

Lemma 5.5. *Let $C_1, \dots, C_d \in \mathbb{Z}_b^{s \times n}$ generate an order 1 digital (v, n, d) -net in base b . Let $\lambda_1, \dots, \lambda_d, \gamma_1, \dots, \gamma_d \in \mathbb{N}_0$. Let $\omega_{\gamma_1, \dots, \gamma_d}^{\lambda_1, \dots, \lambda_d}(\mathfrak{C})$ denote the cardinality of such $t \in \mathfrak{D}(\mathfrak{C})$ with $\varrho_1(t_i) = \gamma_i$ for all $1 \leq i \leq d$ that either $\gamma_i \leq \lambda_i$ or $\varrho_1(t'_i) = \lambda_i$. If $\lambda_1, \dots, \lambda_d \leq s$ then*

$$\omega_{\gamma_1, \dots, \gamma_d}^{\lambda_1, \dots, \lambda_d}(\mathfrak{C}) \leq (b-1)^d b^{(\min(\lambda_1, \gamma_1-1) + \dots + \min(\lambda_d, \gamma_d-1) - n + v)_+}.$$

Proof. Let $t = (t_1, \dots, t_d) \in \mathfrak{D}(\mathfrak{C})$ with $\varrho_1(t_i) = \gamma_i$ for all $1 \leq i \leq d$ and either $\gamma_i \leq \lambda_i$ or $\varrho_1(t'_i) = \lambda_i$. Let t_i have b -adic expansion $t_i = \tau_{i,0} + \tau_{i,1}b + \tau_{i,2}b^2 + \dots$. Let $C_i = (c_{i,1}, \dots, c_{i,s})^\top$, put $\lambda_i^* = \min(\lambda_i, \gamma_i - 1)$ and $c_{i,\gamma_i} = (0, \dots, 0)$ if $\gamma_i > s$, $1 \leq i \leq d$. Then we have

$$\begin{aligned} & c_{1,1}^\top \tau_{1,0} + \dots + c_{1,\lambda_1^*}^\top \tau_{1,\lambda_1^*-1} + c_{1,\gamma_1}^\top \tau_{1,\gamma_1-1} + \\ & \vdots \\ & + c_{d,1}^\top \tau_{d,0} + \dots + c_{d,\lambda_d^*}^\top \tau_{d,\lambda_d^*-1} + c_{d,\gamma_d}^\top \tau_{d,\gamma_d-1} = (0 \dots, 0)^\top \in \mathbb{Z}_b^n. \end{aligned} \tag{14}$$

We put

$$\begin{aligned} A &= (c_{1,1}^\top, \dots, c_{1,\lambda_1^*}^\top, \dots, c_{d,1}^\top, \dots, c_{d,\lambda_d^*}^\top) \in \mathbb{Z}_b^{n \times (\lambda_1^* + \dots + \lambda_d^*)}, \\ y &= (\tau_{1,0}, \dots, \tau_{1,\lambda_1^*-1}, \dots, \tau_{d,0}, \dots, \tau_{d,\lambda_d^*-1})^\top \in \mathbb{Z}_b^{(\lambda_1^* + \dots + \lambda_d^*) \times 1} \end{aligned}$$

and

$$w = -c_{1,\gamma_1}^\top \tau_{1,\gamma_1-1} - \dots - c_{d,\gamma_d}^\top \tau_{d,\gamma_d-1} \in \mathbb{Z}_b^{n \times 1}.$$

Then (14) corresponds to $Ay = w$ and we have

$$\omega_{\gamma_1, \dots, \gamma_d}^{\lambda_1, \dots, \lambda_d}(\mathfrak{C}) = \#\{y \in \mathbb{Z}_b^{\lambda_1^* + \dots + \lambda_d^*} : Ay = w\}.$$

Since C_1, \dots, C_d generate an order 1 digital (v, n, d) -net, the rank of A is $\lambda_1^* + \dots + \lambda_d^*$ if $\lambda_1^* + \dots + \lambda_d^* \leq n - v$. In this case the solution space of the homogeneous system $Ay = (0, \dots, 0)$ has dimension 0. If $\lambda_1^* + \dots + \lambda_d^* > n - v$ then $\text{rank}(A) \geq n - v$ and the dimension of the solution space of the homogeneous system is $\lambda_1^* + \dots + \lambda_d^* - \text{rank}(A) \leq \lambda_1 + \dots + \lambda_d - n + v$. This means that for a given w the system $Ay = w$ has at most 1 solution if $\lambda_1^* + \dots + \lambda_d^* \leq n - v$ and at most $b^{\lambda_1^* + \dots + \lambda_d^* - n + v}$ otherwise. Finally, there are $(b - 1)^d$ possible choices for w since none of the numbers $\tau_{1, \gamma_1 - 1}, \dots, \tau_{d, \gamma_d - 1}$ can be 0. \square

We point out that the condition $\lambda_1, \dots, \lambda_d \leq s$ is not necessary. It just reduces the technicalities but the results would be the same without it. One would have to define $\lambda_i^{**} = \min(\lambda_i^*, s)$ and in the case where $\lambda_i^* > s$ we would get an additional factor $b^{\lambda_i^* - s}$ compensating the restriction.

Proposition 5.6. *Let \mathcal{P}_n^b be an order 1 digital (v, n, d) -net in base b . Let $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$.*

(i) *If $|j|_+ \geq n - v$ then $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \leq b^{-|j|_+ - n + v}$ and $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \leq b^{-2|j|_+}$ for all but at most b^n values of m .*

(ii) *If $|j|_+ < n - v$ then $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \leq b^{-|j|_+ - n + v} (n - v - |j|_+)^{d-1}$.*

Proof. For (i), let $|j|_+ \geq n - v$. Since \mathcal{P}_n^b contains exactly b^n points, there are no more than b^n such m for which $I_{j,m}$ contains a point of \mathcal{P}_n^b meaning that at least all but b^n intervals contain no points at all. Thus the second statement follows from Lemma 5.1. The remaining intervals contain at most b^v points of \mathcal{P}_n^b (Lemma 4.2) so the first statement follows from Lemmas 5.1 and 5.2.

We now prove (ii) so let $|j|_+ < n - v$ and $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$. The function $h_{j,m,l}$ can be given (Lemma 3.3) as

$$h_{j,m,l} = \sum_{\alpha \in \mathbb{N}_0^d} \langle h_{j,m,l}, \text{wal}_\alpha \rangle \text{wal}_\alpha.$$

We apply Lemmas 4.5, 5.3 and 5.4 and get

$$\begin{aligned} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| &= \left| \left\langle \sum_{t \in \mathfrak{D}(\mathfrak{C})} \hat{\chi}_{[0, \cdot)}(t), \sum_{\alpha \in \mathbb{N}_0^d} \langle h_{j,m,l}, \text{wal}_\alpha \rangle \text{wal}_\alpha \right\rangle \right| \\ &\leq \sum_{t \in \mathfrak{D}(\mathfrak{C})} \sum_{\alpha \in \mathbb{N}_0^d} \left| \langle \hat{\chi}_{[0, \cdot)}(t), \text{wal}_\alpha \rangle \right| |\langle h_{j,m,l}, \text{wal}_\alpha \rangle| \end{aligned}$$

$$\begin{aligned}
&\leq b^{-|j|_+} \sum_{t \in \mathfrak{D}(\mathfrak{C})} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ \varrho_1(\alpha_i) = j_i + 1 \\ 1 \leq i \leq d}} \left| \left\langle \hat{\chi}_{[0, \cdot)}(t), \text{wal}_\alpha \right\rangle \right| \\
&\leq b^{-|j|_+} \sum_{t \in \mathfrak{D}(\mathfrak{C})} \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ \alpha_i = t'_i \vee \alpha = t_i \vee \alpha'_i = t_i \\ \varrho_1(\alpha_i) = j_i + 1, 1 \leq i \leq d}} b^{-\max(\varrho_1(\alpha_1), \varrho_1(t_1)) - \dots - \max(\varrho_1(\alpha_d), \varrho_1(t_d))} \\
&= b^{-|j|_+} \sum_{\substack{t \in \mathfrak{D}(\mathfrak{C}) \\ \varrho_1(t_i) \leq j_i + 1 \vee \varrho_1(t'_i) = j_i + 1 \\ 1 \leq i \leq d}} b^{-\max(j_1 + 1, \varrho_1(t_1)) - \dots - \max(j_d + 1, \varrho_1(t_d))} \\
&= b^{-|j|_+} \sum_{\gamma_1, \dots, \gamma_d = 0}^{\infty} b^{-\max(j_1 + 1, \gamma_1) - \dots - \max(j_d + 1, \gamma_d)} \omega_{\gamma_1, \dots, \gamma_d}^{j_1 + 1, \dots, j_d + 1}(\mathfrak{C}). \quad (15)
\end{aligned}$$

By Lemma 5.5 we get

$$\omega_{\gamma_1, \dots, \gamma_d}^{j_1 + 1, \dots, j_d + 1}(\mathfrak{C}) \leq (b - 1)^d b^d$$

since $j_1 + 1, \dots, j_d + 1 \leq n - v \leq s$ and $j_1 + 1 + \dots + j_d + 1 \leq |j|_+ + d < n - v + d$.

We recall that we have $\varrho_1(t) > n - v$ for all $t \in \mathfrak{D}(\mathfrak{C})$. This means that $\omega_{\gamma_1, \dots, \gamma_d}^{j_1 + 1, \dots, j_d + 1}(\mathfrak{C}) = 0$ whenever $\gamma_1 + \dots + \gamma_d \leq n - v$. Therefore $\omega_{\gamma_1, \dots, \gamma_d}^{j_1 + 1, \dots, j_d + 1}(\mathfrak{C}) = 0$ if $\gamma_i \leq j_i$ for all $1 \leq i \leq d$. For any $I \subset \{1, \dots, d\}$ let $I^c = \{1, \dots, d\} \setminus I$. We perform an index shift to get

$$\begin{aligned}
|\langle D_{\mathcal{P}_n^b}, h_{j, m, l} \rangle| &\leq b^{-|j|_+} \sum_{\substack{\gamma_1, \dots, \gamma_d = 0 \\ \gamma_1 + \dots + \gamma_d > n - v}}^{\infty} b^{-\max(j_1 + 1, \gamma_1) - \dots - \max(j_d + 1, \gamma_d)} \\
&= b^{-|j|_+} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1} + 1)} \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} \sum_{\substack{\gamma_{i_2} \geq j_{i_2} + 1 \\ i_2 \in I^c}} b^{-\sum_{\kappa_2 \in I^c} \gamma_{\kappa_2}} \\
&\quad \gamma_1 + \dots + \gamma_d \geq \max \left(n - v + 1, \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1) \right) \\
&= b^{-|j|_+} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1} + 1) - \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1)} \dots \\
&\quad \dots \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} \sum_{\substack{\gamma_{i_2} \geq 0, i_2 \in I^c}} b^{-\sum_{\kappa_2 \in I^c} \gamma_{\kappa_2}} \\
&\quad \sum_{\kappa_2 \in I^c} \gamma_{\kappa_2} \geq \left(n - v - \sum_{\kappa_1 \in I} \gamma_{\kappa_1} - \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1) + 1 \right)_+ \\
&\leq b^{-|j|_+} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1} + 1) - \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1)} \dots
\end{aligned}$$

$$\begin{aligned}
& \dots \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} \sum_{r=\left(n-v-\sum_{\kappa_1 \in I} \gamma_{\kappa_1} - \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1)+1\right)_+}^{\infty} b^{-r} (r+1)^{d-1-\#I} \\
& \leq b^{-|j|_+} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1}+1) - \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1)} \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} b^{-n+v+\sum_{\kappa_1 \in I} \gamma_{\kappa_1} + \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1)} \times \\
& \quad \times \left(n - v - \sum_{\kappa_1 \in I} \gamma_{\kappa_1} - \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1) + 1 \right)_+^{d-1-\#I} \\
& \leq b^{-|j|_+-n+v} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1}+1)} \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} b^{\sum_{\kappa_1 \in I} \gamma_{\kappa_1}} \times \\
& \quad \times \left(n - v - \sum_{\kappa_1 \in I} \gamma_{\kappa_1} - \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1) + 1 \right)_+^{d-1} \\
& \leq b^{-|j|_+-n+v} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1}+1)} \sum_{\kappa_1 \in I} b^{(j_{\kappa_1}+1)} \\
& \quad \times \left(n - v - \sum_{\kappa_1 \in I} (j_{\kappa_1} + 1) - \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1) + 1 \right)_+^{d-1} \\
& \leq b^{-|j|_+-n+v} (n - v - |j|_+)^{d-1}.
\end{aligned}$$

□

Proposition 5.7. Let \mathcal{P}_n^b be an order 2 digital (v, n, d) -net in base b . Let $j \in \mathbb{N}_{-1}^d$, $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$.

(i) If $|j|_+ \geq n - \lceil v/2 \rceil$ then $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \leq b^{-|j|_+-n+v/2}$ and $|\mu_{j,m,l}(D_{\mathcal{P}_n^b})| \leq b^{-2|j|_+}$ for all but b^n values of m .

(ii) If $|j|_+ < n - \lceil v/2 \rceil$ then $|\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \leq b^{-2n+v} (2n - v - 2|j|_+)^{d-1}$.

Proof. According to Lemma 4.1, \mathcal{P}_n^b is an order 1 digital $(\lceil v/2 \rceil, n, d)$ -net. Hence (i) follows from Proposition 5.6.

We now prove (ii) so let $|j|_+ < n - \lceil v/2 \rceil$ and $m \in \mathbb{D}_j$, $l \in \mathbb{B}_j$. We start at (15) and recall that we have $\varrho_2(t) > 2n - v$ for all $t \in \mathfrak{D}(\mathfrak{C})$. This means that $\omega_{\gamma_1, \dots, \gamma_d}^{j_1+1, \dots, j_d+1}(\mathfrak{C}) = 0$ whenever $\gamma_1 + \min(\gamma_1, j_1 + 1) + \dots + \gamma_d + \min(\gamma_d, j_d + 1) \leq 2n - v$. We argue similarly

to the proof of Proposition 5.6 to get

$$\begin{aligned}
& |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle| \preceq b^{-|j|+} \sum_{\gamma_1, \dots, \gamma_d=0}^{\infty} b^{-\max(j_1+1, \gamma_1) - \dots - \max(j_d+1, \gamma_d)} \omega_{\gamma_1, \dots, \gamma_d}^{j_1+1, \dots, j_d+1} \\
& \preceq b^{-|j|+} \sum_{\substack{\gamma_1, \dots, \gamma_d=0 \\ \gamma_1 + \min(\gamma_1, j_1+1) + \dots + \gamma_d + \min(\gamma_d, j_d+1) > 2n-v}}^{\infty} b^{-\max(j_1+1, \gamma_1) - \dots - \max(j_d+1, \gamma_d)} \\
& = b^{-|j|+} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1}+1)} \dots \\
& \quad \dots \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} \sum_{\substack{\gamma_{i_2} \geq j_{i_2}+1 \\ i_2 \in I^c}} b^{-\sum_{\kappa_2 \in I^c} \gamma_{\kappa_2}} \\
& \quad \quad \quad 2 \sum_{\kappa_1 \in I} \gamma_{\kappa_1} + \sum_{\kappa_2 \in I^c} (\gamma_{\kappa_2} + j_{\kappa_2} + 1) \geq \max \left(2n-v+1, 2 \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1) \right) \\
& = b^{-|j|+} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1}+1) - \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1)} \dots \\
& \quad \dots \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} \sum_{\substack{\gamma_{i_2} \geq 0, i_2 \in I^c}} b^{-\sum_{\kappa_2 \in I^c} \gamma_{\kappa_2}} \\
& \quad \quad \quad \sum_{\kappa_2 \in I^c} \gamma_{\kappa_2} \geq \left(2n-v-2 \sum_{\kappa_1 \in I} \gamma_{\kappa_1} - 2 \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1) + 1 \right)_+ \\
& \leq b^{-|j|+} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1}+1) - \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1)} \dots \\
& \quad \dots \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} \sum_{r=\left(2n-v-2 \sum_{\kappa_1 \in I} \gamma_{\kappa_1} - 2 \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1) + 1 \right)_+}^{\infty} b^{-r(r+1)^{d-1-\#I}} \\
& \leq b^{-|j|+} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1}+1) - \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1)} \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} b^{-2n+v+2 \sum_{\kappa_1 \in I} \gamma_{\kappa_1} + 2 \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1)} \times \\
& \quad \times \left(2n-v-2 \sum_{\kappa_1 \in I} \gamma_{\kappa_1} - 2 \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1) + 1 \right)^{d-1-\#I} \\
& \leq b^{-|j|+2n+v} \sum_{I \subsetneq \{1, \dots, d\}} b^{-\sum_{\kappa_1 \in I} (j_{\kappa_1}+1) + \sum_{\kappa_2 \in I^c} (j_{\kappa_2}+1)} \sum_{\substack{0 \leq \gamma_{i_1} \leq j_{i_1} \\ i_1 \in I}} b^{2 \sum_{\kappa_1 \in I} \gamma_{\kappa_1}} \times
\end{aligned}$$

$$\begin{aligned}
& \times \left(2n - v - 2 \sum_{\kappa_1 \in I} \gamma_{\kappa_1} - 2 \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1) + 1 \right)^{d-1} \\
& \leq b^{-|j|_+ - 2n + v} \sum_{I \subsetneq \{1, \dots, d\}} b^{\sum_{\kappa_1 \in I} (j_{\kappa_1} + 1) + \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1)} \times \\
& \quad \times \left(2n - v - 2 \sum_{\kappa_1 \in I} (j_{\kappa_1} + 1) - 2 \sum_{\kappa_2 \in I^c} (j_{\kappa_2} + 1) + 1 \right)^{d-1} \\
& \leq b^{-2n+v} (2n - v - 2|j|_+)^{d-1}.
\end{aligned}$$

□

We are now ready to prove the theorems.

Proof of Theorem 1.1. Let $D_{\mathcal{P}_n^b}$ be an order 2 digital (v, n, d) -net in base b . We apply Theorem 3.1, hence we need to prove

$$\sum_{j \in \mathbb{N}_{-1}^d} b^{|j|_+} \sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle|^2 \leq b^{-2n+v} n^{d-1} v.$$

We recall that $\#\mathbb{D}_j = b^{|j|_+}$, $\#\mathbb{B}_j = b - 1$. We split the sum and apply Proposition 5.7 to get

$$\begin{aligned}
& \sum_{j \in \mathbb{N}_{-1}^d} b^{|j|_+} \sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle|^2 \\
& \stackrel{|j|_+ < n - \lceil v/2 \rceil}{\leq} \sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ < n - \lceil v/2 \rceil}} b^{|j|_+} b^{|j|_+} b^{-4n+2v} (2n - v - 2|j|_+)^{2(d-1)} \\
& \leq b^{-4n+2v} \sum_{\kappa=0}^{n-v/2-1} b^{2\kappa} (2n - v - 2\kappa)^{2(d-1)} (\kappa + 1)^{d-1} \\
& \leq b^{-4n+2v} b^{2n-v} (2n - v - 2n + v)^{2(d-1)} (n - v/2)^{d-1} \\
& \leq b^{-2n+v} n^{d-1}
\end{aligned}$$

for big intervals,

$$\sum_{\substack{j \in \mathbb{N}_{-1}^d \\ n > |j|_+ \geq n - \lceil v/2 \rceil}} b^{|j|_+} \sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle|^2$$

$$\begin{aligned}
&\preceq \sum_{\substack{j \in \mathbb{N}_{-1}^d \\ n > |j|_+ \geq n - \lceil v/2 \rceil}} b^{|j|_+} b^{|j|_+} b^{-2|j|_+ - 2n + v} \\
&\leq b^{-2n+v} \sum_{\kappa = n - \lceil v/2 \rceil}^{n-1} (\kappa + 1)^{d-1} \\
&\preceq b^{-2n+v} n^{d-1} v
\end{aligned}$$

for middle intervals and

$$\begin{aligned}
&\sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ \geq n}} b^{|j|_+} \sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle|^2 \\
&\preceq \sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ \geq n}} b^{|j|_+} b^n b^{-2|j|_+ - 2n + v} + \sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ \geq n}} b^{|j|_+} (b^{|j|_+} - b^n) b^{-4|j|_+} \\
&\leq b^{-n+v} \sum_{\kappa = n}^{\infty} b^{-\kappa} (\kappa + 1)^{d-1} + \sum_{\kappa = n}^{\infty} b^{-2\kappa} (\kappa + 1)^{d-1} \\
&\preceq b^{-2n+v} n^{d-1}
\end{aligned}$$

for small intervals. □

Proof of Theorem 1.2. Let $D_{\mathcal{P}_n^b}$ be an order 1 digital (v, n, d) -net in base b . We apply Theorem 3.2, hence we need to prove

$$\sum_{j \in \mathbb{N}_{-1}^d} b^{|j|_+ + (r-1/p+1)q} \left(\sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle|^p \right)^{q/p} \preceq b^{n(r-1)q} n^{(d-1)q} b^{vq}.$$

We recall that $\#\mathbb{D}_j = b^{|j|_+}$, $\#\mathbb{B}_j = b - 1$. We split the sum and apply Minkowski's inequality and Proposition 5.6 to get

$$\begin{aligned}
&\sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ < n-v}} b^{|j|_+ + (r-1/p+1)q} \left(\sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle|^p \right)^{q/p} \\
&\preceq \sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ < n-v}} b^{|j|_+ + (r-1/p+1)q} b^{|j|_+ + q/p} b^{(-|j|_+ - n + v)q} (n - v - |j|_+)^{(d-1)q} \\
&\leq b^{(-n+v)q} \sum_{\kappa=0}^{n-v-1} b^{\kappa r q} (n - v - \kappa)^{(d-1)q} (\kappa + 1)^{d-1}
\end{aligned}$$

$$\begin{aligned}
&\leq b^{(-n+v)q} b^{(n-v)rq} (n-v+1)^{d-1} \\
&\preceq b^{n(r-1)q} n^{d-1} b^{v(1-r)q}
\end{aligned}$$

for big intervals,

$$\begin{aligned}
&\sum_{\substack{j \in \mathbb{N}_{-1}^d \\ n > |j|_+ \geq n-v}} b^{|j|_+(r-1/p+1)q} \left(\sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle|^p \right)^{q/p} \\
&\preceq \sum_{\substack{j \in \mathbb{N}_{-1}^d \\ n > |j|_+ \geq n-v}} b^{|j|_+(r-1/p+1)q} b^{|j|_+q/p} b^{(-|j|_+-n+v)q} \\
&\leq b^{(-n+v)q} \sum_{\kappa=n-v}^{n-1} b^{\kappa r q} (\kappa+1)^{d-1} \\
&\preceq b^{(-n+v)q} b^{nrq} n^{d-1} \\
&\leq b^{n(r-1)q} n^{(d-1)} b^{vq}
\end{aligned}$$

for middle intervals and considering the range of r

$$\begin{aligned}
&\sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ \geq n}} b^{|j|_+(r-1/p+1)q} \left(\sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle|^p \right)^{q/p} \\
&\preceq \sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ \geq n}} b^{|j|_+(r-1/p+1)q} b^{nq/p} b^{(-|j|_+-n+v)q} + \sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ \geq n}} b^{|j|_+(r-1/p+1)q} (b^{|j|_+} - b^n)^{q/p} b^{-2|j|_+q} \\
&\leq b^{nq/p} b^{(-n+v)q} \sum_{\kappa=n}^{\infty} b^{\kappa(r-1/p)q} (\kappa+1)^{d-1} + \sum_{\kappa=n}^{\infty} b^{\kappa(r-1)q} (\kappa+1)^{d-1} \\
&\preceq b^{nq/p} b^{(-n+v)q} b^{n(r-1/p)q} n^{d-1} + b^{n(r-1)q} n^{d-1} \\
&\preceq b^{n(r-1)q} n^{(d-1)} b^{vq}
\end{aligned}$$

for small intervals. □

Proof of Theorem 1.3. Let $D_{\mathcal{P}_n^b}$ be an order 2 digital (v, n, d) -net in base b . The proof is similar to the proof of Theorem 1.2. We apply Proposition 5.7 instead of 5.6 to get

$$\sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ < n - \lceil v/2 \rceil}} b^{|j|_+(r-1/p+1)q} \left(\sum_{m \in \mathbb{D}_j, l \in \mathbb{B}_j} |\langle D_{\mathcal{P}_n^b}, h_{j,m,l} \rangle|^p \right)^{q/p}$$

$$\begin{aligned}
&\preceq \sum_{\substack{j \in \mathbb{N}_{-1}^d \\ |j|_+ < n - \lceil v/2 \rceil}} b^{|j|_+ + (r-1/p+1)q} b^{|j|_+ + q/p} b^{(-2n+v)q} (2n - v - 2|j|_+)^{(d-1)q} \\
&\leq b^{(-2n+v)q} \sum_{\kappa=0}^{n-v/2-1} b^{\kappa(r+1)q} (2n - v - 2\kappa)^{(d-1)q} (\kappa + 1)^{d-1} \\
&\leq b^{(-2n+v)q} b^{(n-v/2)(r+1)q} (n - v/2 + 1)^{d-1} \\
&\preceq b^{n(r-1)q} n^{d-1} b^{v/2(1-r)q}
\end{aligned}$$

and analogous results for the other subsums. \square

Proof of Corollaries 1.4, 1.5, 1.6 and 1.7. The results for the Triebel-Lizorkin spaces follow from Theorem 2.1 and Theorems 1.2 and 1.3, respectively. The results for the Sobolev spaces then follow in the case $q = 2$. \square

Proof of Theorem 1.8. The result follows from Corollary 1.7 in the case $r = 0$. \square

References

- [B11] D. Bilyk, *On Roth's orthogonal function method in discrepancy theory*. Unif. Distrib. Theory **6** (2011), 143–184.
- [BLV08] D. Bilyk, M. T. Lacey, A. Vagharshakyan, *On the small ball inequality in all dimensions*. J. Funct. Anal. **254** (2008), 2470–2502.
- [BTY12] D. Bilyk, V. N. Temlyakov, R. Yu, *Fibonacci sets and symmetrization in discrepancy theory*. J. Complexity **28** (2012), 18–36.
- [C80] W. W. L. Chen, *On irregularities of distribution*. Mathematika **27** (1981), 153–170.
- [CS02] W. W. L. Chen, M. M. Skrikanov, *Explicit constructions in the classical mean squares problem in irregularities of point distribution*. J. Reine Angew. Math. **545** (2002), 67–95.
- [CS08] W. W. L. Chen, M. M. Skrikanov, *Orthogonality and digit shifts in the classical mean squares problem in irregularities of point distribution*. In: Diophantine approximation, 141–159, Dev. Math., **16**, Springer, Vienna, 2008.

- [D56] H. Davenport, *Note on irregularities of distribution*. Mathematika **3** (1956), 131–135.
- [D07] J. Dick, *Explicit constructions of quasi-Monte Carlo rules for the numerical integration of high-dimensional periodic functions*. SIAM J. Numer. Anal. **45** (2007), 2141–2176.
- [D14] J. Dick, *Discrepancy bounds for infinite-dimensional order two digital sequences over \mathbb{F}_2* . J. Number Theory **136** (2014), 204–232.
- [DP10] J. Dick, F. Pillichshammer, Digital nets and sequences. Discrepancy theory and quasi-Monte Carlo integration. Cambridge University Press, Cambridge, 2010.
- [DP14a] J. Dick, F. Pillichshammer, *Optimal \mathcal{L}_2 discrepancy bounds for higher order digital sequences over the finite field \mathbb{F}_2* . Acta Arith. **162** (2014), 65–99.
- [DP14b] J. Dick, F. Pillichshammer, *Explicit constructions of point sets and sequences with low discrepancy*. To appear in P. Kritzer, H. Niederreiter, F. Pillichshammer, A. Winterhof, Uniform distribution and quasi-Monte Carlo methods - Discrepancy, Integration and Applications (2014).
- [FPPS10] H. Faure, F. Pillichshammer, G. Pirsic, W. Ch. Schmid, *L_2 discrepancy of generalized two-dimensional Hammersley point sets scrambled with arbitrary permutations*. Acta Arith. **141** (2010), 395–418.
- [H81] G. Halász, On Roth’s method in the theory of irregularities of point distributions. Recent progress in analytic number theory, Vol. 2, 79–94. Academic Press, London-New York, 1981.
- [Hi10] A. Hinrichs, *Discrepancy of Hammersley points in Besov spaces of dominating mixed smoothness*. Math. Nachr. **283** (2010), 478–488.
- [Hi14] A. Hinrichs, *Discrepancy, Integration and Tractability*. In J. Dick, F. Y. Kuo, G. W. Peters, I. H. Sloan, Monte Carlo and Quasi-Monte Carlo Methods 2012 (2014).
- [HM11] A. Hinrichs, L. Markhasin, *On lower bounds for the L_2 -discrepancy*. J. Complexity **27** (2011), 127–132.
- [KN74] L. Kuipers, H. Niederreiter, Uniform distribution of sequences. John Wiley & Sons, Ltd., New York, 1974.

- [M13a] L. Markhasin, *Discrepancy of generalized Hammersley type point sets in Besov spaces with dominating mixed smoothness*. Unif. Distrib. Theory **8** (2013), 135–164.
- [M13b] L. Markhasin, *Quasi-Monte Carlo methods for integration of functions with dominating mixed smoothness in arbitrary dimension*. J. Complexity **29** (2013), 370–388.
- [M13c] L. Markhasin, *Discrepancy and integration in function spaces with dominating mixed smoothness*. Dissertationes Math. **494** (2013), 1–81.
- [M99] J. Matoušek, *Geometric discrepancy. An illustrated guide*. Springer-Verlag, Berlin, 1999.
- [N87] H. Niederreiter, G. Pirsic, *Duality for digital nets and its applications*. Acta Arith. **97** (2001), 173–182.
- [NP01] H. Niederreiter, *Point sets and sequences with small discrepancy*. Monatsh. Math. **104** (1987), 273–337.
- [NW10] E. Novak, H. Woźniakowski, *Tractability of multivariate problems. Volume II: Standard information for functionals*. European Mathematical Society Publishing House, Zürich, 2010.
- [R54] K. F. Roth, *On irregularities of distribution*. Mathematika **1** (1954), 73–79.
- [R80] K. F. Roth, *On irregularities of distribution. IV*. Acta Arith. **37** (1980), 67–75.
- [S72] W. M. Schmidt, *Irregularities of distribution. VII*. Acta Arith. **21** (1972), 45–50.
- [S77] W. M. Schmidt, *Irregularities of distribution X. Number Theory and Algebra*, 311–329. Academic Press, New York, 1977.
- [S06] M. M. Skriganov, *Harmonic analysis on totally disconnected groups and irregularities of point distributions*. J. Reine Angew. Math. **600** (2006), 25–49.
- [T10] H. Triebel, *Bases in function spaces, sampling, discrepancy, numerical integration*. European Mathematical Society Publishing House, Zürich, 2010.